A counterexample to a conjectured entanglement inequality

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We give an explicit counterexample to an entanglement inequality suggested in a recent paper [quant-ph/0005126] by Benatti and Narnhofer. The inequality would have had far-reaching consequences, including the additivity of the entanglement of formation.

The problem of additivity of entanglement of formation [1] is one of the fundamental open issues in the theory of entanglement. In a recent paper [2] Fabio Benatti and Heide Narnhofer bring to bear on this problem their intuition gained from a completely different enterprise (the study of quantum dynamical entropy), using the equivalence between the entanglement of formation and a quantity called the "entropy of a subalgebra" with respect to a state (a connection noted earlier by Uhlmann [3] and others). They achieve some interesting partial results supporting the additivity conjecture[1].

In this brief note we will concentrate on an exciting prospect coming up in their paper as inequality (12) (restated below as equation (2)). Benatti and Narnhofer apparently found it as an inequality one would just love to have for a simple proof of additivity, but seem undecided as to its validity. Given the simplicity and generality of the inequality, probably many more results would follow from it. It therefore seemed necessary to us to decide the issue quickly. Unfortunately, (if one can ever say that of a mathematical statement) the inequality turned out to be false in general.

The inequality in question refers to a vector Ψ in a fourfold tensor product $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$, of which the factors $\mathcal{H}_1$ and $\mathcal{H}_3$ are associated with one party (say Alice) and factors $\mathcal{H}_2$ and $\mathcal{H}_4$ are associated with another, Bob. We can write Ψ in Schmidt form as

$$\Psi = \sum_\alpha \sqrt{\lambda_\alpha}\, \Phi_\alpha^{12} \otimes \Phi_\alpha^{34} \tag{1}$$

with respect to another split 12|34, of the system, which is the split according to which additivity would be investigated. Of course, by definition of the Schmidt decomposition, we have $\sum_\alpha \lambda_\alpha = 1$, and the vectors $\Phi_\alpha^{12} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\Phi_\alpha^{34} \in \mathcal{H}_3 \otimes \mathcal{H}_4$ each form an orthonormal set. Let us denote by $S(\rho)$ the von Neumann entropy of a density operator ρ, and by $\mathrm{tr}_i(A)$ (resp. $\mathrm{tr}_{ij}(A)$) the partial trace of the operator A with respect to the Hilbert space $\mathcal{H}_i$ (resp. $\mathcal{H}_i \otimes \mathcal{H}_j$), assumed to be a tensor factor of the space on which A lives. Then the inequality suggested by Benatti and Narnhofer is

$$S\Big(\mathrm{tr}_{24}(|\Psi\rangle\langle\Psi|)\Big) \geq \sum_\alpha \lambda_\alpha \Big\{ S\big(\mathrm{tr}_2(|\Phi_\alpha^{12}\rangle\langle\Phi_\alpha^{12}|)\big) + +S\big(\mathrm{tr}_4(|\Phi_\alpha^{34}\rangle\langle\Phi_\alpha^{34}|)\big) \Big\}\,. \tag{2}$$

A superficial random numerical test might come out in favor of (2). A counterexample is found, however, with all Hilbert spaces $\mathcal{H}_i = \mathbb{C}^d$, d arbitrary, namely

$$\Psi = \frac{1}{d} \sum_{i,k=1}^{d} e_i \otimes e_k \otimes e_i \otimes e_k = \frac{1}{d} \sum_{\alpha=1}^{d^2} \Phi_\alpha \otimes \overline{\Phi_\alpha}\,, \tag{3}$$

where the vectors e_i form an orthonormal basis of $\mathbb{C}^d$. Thus Ψ is the tensor product of the vector $d^{-1/2}\sum_i e_i \otimes e_i$ for Alice and the same vector for Bob, making the left hand side of inequality (2) zero. Ψ is also maximally entangled for the split 12|34. One Schmidt decomposition is into the vectors $e_i \otimes e_k$ (which would make the right hand side of (2) zero, too). But in the maximally entangled case the Schmidt decomposition is highly non-unique, and we may also choose another decomposition using entangled vectors Φ_α and their complex conjugates $\overline{\Phi_\alpha}$ with respect to the basis e_i. We may even take them maximally entangled [4], whence the right hand side of (2) becomes $2\log d$. By deforming the coefficients in the latter decomposition slightly, we also get examples with unique Schmidt decomposition such that LHS(2)≈ 0 and RHS(2)$\approx 2\log d$.

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[1] We only mention in passing an omission in the formulation of Proposition 1 in [2]: the unitary U must be assumed to factorize.

[1] C.H. Bennett et al, *Phys.Rev. A* **54** (1996)3824, quant-ph/9604024

[2] F. Benatti and H. Narnhofer: "On the additivity of the entanglement of formation", quant-ph/0005126.

[3] A. Uhlmann:"Entropy and optimal decompositions of states relative to a maximal commutative subalgebra", quant-ph/9704017

[4] K.G.H. Vollbrecht and R.F. Werner: Why two qubits are special, quant-ph/9910064.